Lecture No. 1

Introduction to Method of Weighted Residuals

- Solve the differential equation $L(u) = p(x)$ in V

  where L is a differential operator

  with boundary conditions $S(u) = g(x)$ on Γ

  where S is a differential operator

- Find an approximation, u_{app}, which satisfies the above equation

$$u_{app} = u_B + \sum_{k=1}^{N} \alpha_k \, \phi_k(x)$$

  where α_k = unknown parameters which we must find

  ϕ_k = set of known functions which we define a priori

- The approximating functions that make up u_{app} must be selected such that they satisfy:
  - ***Admissibility conditions:*** these define a set of entrance requirements.
  - ***Completeness:*** ensures that the procedure will work.

Basic Definitions

1. Admissibility of functions

In order for a function to be admissible a function must

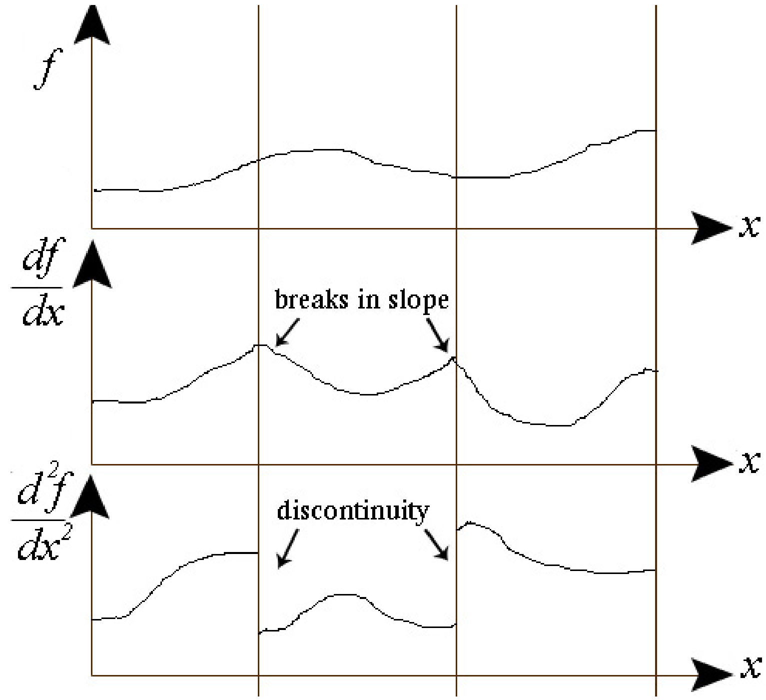
- Satisfy the specified boundary conditions
- Be continuous such that interior domain functional continuity requirements are satisfied

Thus for a function f to be admissible for our stated problem we must have:

- Boundary conditions satisfied $\Rightarrow S(f)=g(x)$ on Γ
- f must have the correct degree of functional continuity

  e.g. to satisfy

  $L(f) = \frac{d^2f}{dx^2}$, the function and its first derivative must be continuous.



This defines the Sobelov Space requirements (used to describe functional continuity).

Relaxed admissibility conditions: we may back off from some of the stated admissibility conditions – either which b.c.'s we satisfy or what degree of functional continuity we require

2. Measure of a Function

- Point Norm defines the maximum value and as such represents a point measure of a function

  - Point norm of vector $\underline{a} \rightarrow$ maximum element of $\underline{a} \rightarrow a_{\max}$

    therefore we select the max values of $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

  - Point norm of a function $f \rightarrow$ maximum value of f within the domain $\rightarrow f_{\max}$

- Euclidian Norm represents an integral measure:

  - The magnitude of a vector may also be expressed as:

$$|\underline{a}|^2 = a_1^2 + a_2^2 + \cdots$$

$$|\underline{a}| = \left[\underline{a}^T \underline{a}\right]^{1/2}$$

This represents the inner produce of the vector onto itself. Note that the mean square value represents an integral measure as well.

- Integral measure of a function

Let's extend the idea of a norm back to an integral when an infinite number of values between x_1 and x_2 occur.

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_n \end{bmatrix} \Rightarrow n \to \infty$$

Therefore there are an infinite number of elements in the vector.

This can be represented by the segment $x_1 < x < x_2$.

The integral norm of the functional values over the segment is defined as:

$$\|f\|_E^2 = \int_{x_1}^{x_2} f^2 dx$$

We use a double bar for the Euclidian Norm to distinguish it from a point norm.

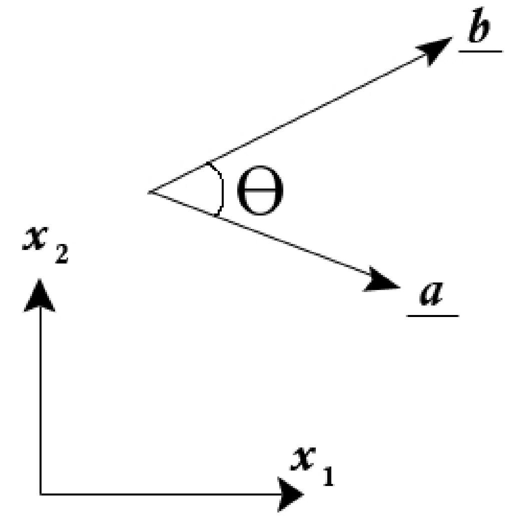
Note that $\|f\|_E \geq 0$ and only equals zero when $f = 0$. Therefore, we can use norms as a measure of how well our approximation to the solution is doing (e.g. examine $\|u_{app} - u\|$)

We'll be using Euclidian norms.

3. Orthogonality of a function

- We use orthogonality as a filtering process in the selection of functions and in driving the error to zero.

  Vectors are orthogonal when $\Theta = 90°$



- A test for orthogonality is the dot product or <u>inner product</u>:

  $a \cdot b = |a||b| \cos \Theta = a_1 b_1 + a_2 b_2$

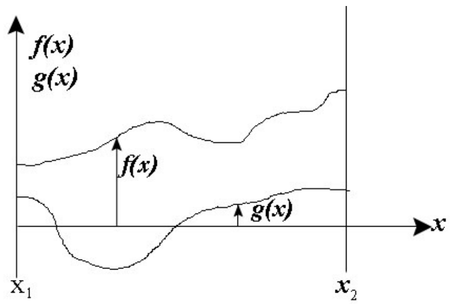
  where

$$\underline{a} = a_1 \hat{\imath}_1 + a_2 \hat{\imath}_2$$

$$\underline{a} \cdot \underline{b} = \underline{a}^T \underline{b} = \underline{b}^T \underline{a}$$

Hence if $\underline{a} \cdot \underline{b} = 0$, vectors $\underline{a}$ and $\underline{b}$ are orthogonal. This concept can now be extended to *N* dimensions.

- Extend the vector definitions of orthogonality to the limit as $N \to \infty$ (i.e. to functions)



Examine $\int_{x_1}^{x_2} f \cdot g dx$

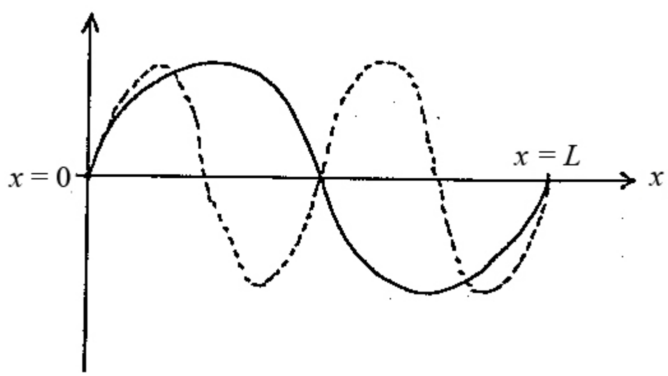
If this equals zero, then the functions are orthogonal.

Therefore orthogonality of functions depends on both the interval and the functions.

- The inner product of 2 functions establishes the condition of orthogonality:

$$\int_{x_1}^{x_2} f \cdot g \, dx = \langle f, g \rangle$$

e.g. $\sin \frac{n\pi x}{L} n = 0, 1, 2 \ldots$ defines a set of functions which are orthogonal over the interval $[0, L]$. The figure shows two such functions which are orthogonal over this interval:



In addition $\sin \frac{n\pi x}{L}$ functions vanish at the ends of the interval. This is a useful feature.

- For real functions:

$$< \phi_1, \phi_2 > = < \phi_2, \phi_1 >$$

$$\alpha < \phi_1, \phi_2 > = < \alpha\, \phi_1, \phi_2 >$$

$$< \phi_1, \phi_2 + \phi_3 > = < \phi_1, \phi_2 > + < \phi_1, \phi_3 >$$

- Linear Independence: A sequence of functions $\phi_1 (x), \phi_2 (x), \ldots, \phi_n (x)$ is linearly independent if:

$$\alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \ldots + \alpha_n \phi_n = 0$$

for any point x within the interval only when $\alpha_i = 0$ for all i.

An orthogonal set will be linearly independent.

4. Completeness

- Consider *n* functions $\phi_1, \phi_2, \ldots, \phi_n$ which are admissible. Therefore they satisfy functional continuity and the specified b.c.'s. In addition these functions are linearly independent.

- Now set up the approximate solution:
  - ***A sequence of linearly independent functions is said to be complete if we have convergence as $N \to \infty$.***

    Therefore functions comprise a complete sequence if $\|u - u_{app}\| \to 0$ as $N \to \infty$ where u = the exact solution and u_{app} = our approximate solution.

    Hence we require convergence of the norm.

- Examples of complete sequences:
  - Sines
  - Polynomials
  - Bessel functions

Summary of Basic Definitions

1. Admissibility: these represent our entrance requirements.
2. Norm: indicates how we measure things
3. Orthogonality: allows us to drive the error to zero.
4. Completeness: tells us if it will work?

Solution Procedure

Given:

$$L(u) = p(x) \quad \text{in } V$$

$$S(u) = g(x) \quad \text{on } \Gamma$$

We define an approximate solution in series form

$$u_{app} = u_B + \sum_{k=1}^{N} \alpha_k \phi_k$$

where

α_k are unknown parameters

ϕ_k are a set of known functions from a complete sequence

- We must enforce ***admissibility***
  - ***Boundary condition satisfaction:***

    Ensure that $S(u_{app}) = g$ on Γ

    Let's pick u_B such that

    $$S(u_B) = g \quad \text{on} \quad \Gamma$$

    Since u_B satisfied the b.c.'s, all ϕ_k must vanish on the boundary

    $$S(\phi_k) = 0 \;\; \forall \;\; k$$

    Thus each ϕ_k must individually vanish on the boundary.

  - In addition all ϕ_k's satisfy the ***functional continuity requirements***, they form an admissible set of functions.

- So far we have enforced satisfaction of u_{app} on the boundary. However we violate the d.e. in the interior.

  This defines the Residual Error.

$$\mathcal{E}_I = L(u_{app}) - p(x)$$

$$\Rightarrow$$

$$\mathcal{E}_I = L(u_B) + \sum_{k=1}^{N} \alpha_k L(\phi_k) - p(x)$$

We note that $\mathcal{E}_I$ represents a point measure of the interior error.

For the exact solution, $\mathcal{E}_I = 0 \; \forall \; x$ in V

- We must solve for N different unknown coefficients, α_k , $k = 1, N$.

  To accomplish this we select N different independent functions $w_1, w_2, w_3 \ldots w_N$ ***and let:***

$$\int_v \mathcal{E}_I \, w_i dx = < \mathcal{E}_I, w_i > = 0 \text{ for } i = 1, 2, \ldots N$$

  Therefore we constrain the inner product of the error and a set of weighting functions to be zero.

  Note: if we don't select w_i, $i = 1, N$ functions to be linearly independent, we'll get duplicate equations and ultimately generate a singular matrix.

- Hence we have posed N constraints on the residual

  ϕ_i's are designated as the trial functions

  w_i's are designated as the test functions (they test how good the solution is).

- Substituting for $\mathcal{E}_I$ into the integral inner product relationship

$$\sum_{k=1}^{N} \alpha_k < w_i, L(\phi_k) > = \; -< (L(u_B) - p, w_i) > \qquad\qquad i = 1, 2, \dots N$$

We define

$$\boldsymbol{a}_{i,k} \equiv \; \langle w_i, L(\phi_k) \rangle$$

$$c_i \equiv \; -\langle L(u_B) - p, w_i \rangle$$

Thus we can write the system of simultaneous algebraic equations

$$\sum_{k=1}^{N} \boldsymbol{a}_{i,k} \; \alpha_k = \; c_i \qquad i = 1, 2, \dots N$$

We note that

k = column index; i = row index

Hence we now have a set of algebraic equations from our d.e.

$$\boldsymbol{a}_{i,k} \, \alpha_k = c_i$$

and we can solve for our unknowns, α_k.

- In the operator $L(u)$ is linear, we get N linear algebraic equations. When the d.e. is nonlinear, the method still works but you get nonlinear algebraic equations.

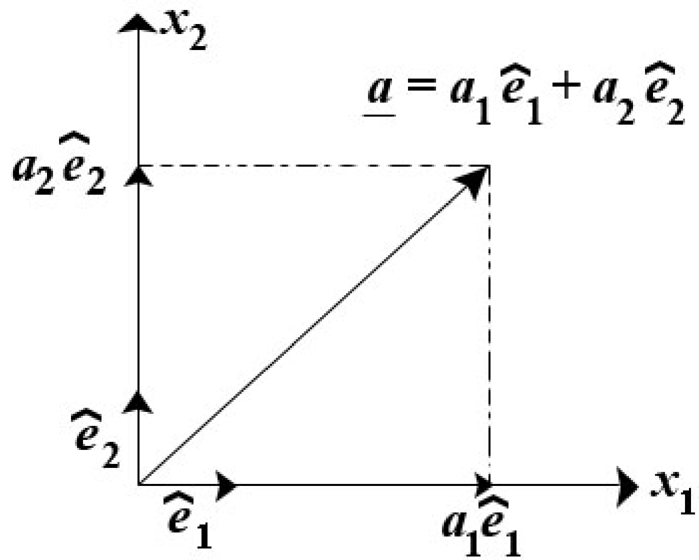
- Then we require the test functions w_i to be orthogonal to the residual, since

$$< \mathcal{E}_I, w_i > = 0$$

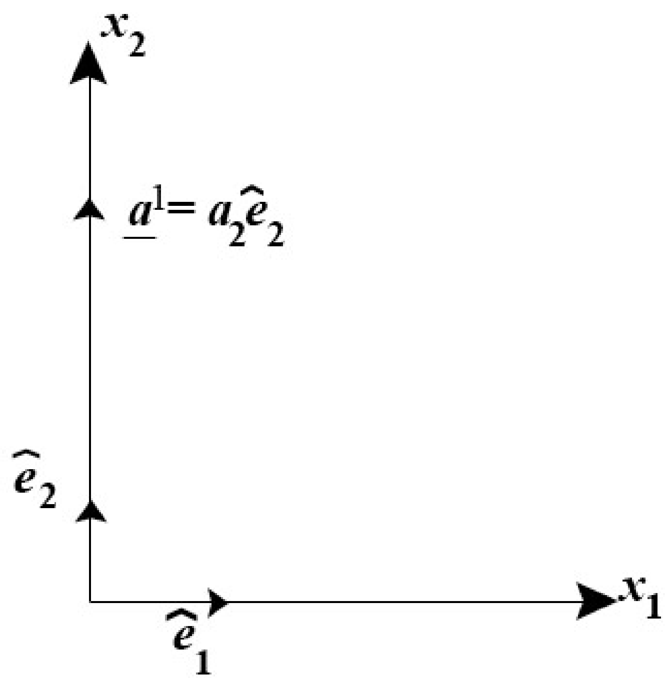
In the limit we would require an infinite number of test functions to be orthogonal to the residual. In the limit, the error diminishes to zero.

Analogy to vectors

- Let some vector $\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2$ represent the error. Thus the coefficients of the vector a_1 and a_2 represent components of some error. $\hat{e}_1$ and $\hat{e}_2$ are the unit directions and also represent the test functions which are orthogonal and linearly independent.



- Now let's constrain $\underline{a}$ such that $\underline{a} \cdot \hat{e}_1 = 0$. This constrains $\underline{a}$ such that $a_1 = 0$.
- Now select another vector independent of $\hat{e}_1$. We therefore select $\hat{e}_2$ for the next orthogonality constraint.



Therefore we now force $\underline{a} \cdot \hat{e}_2 = 0 \Rightarrow (a_2\,\hat{e}_2) \cdot\, \hat{e}_2 = 0$ and thus we constrain $a_2 = 0$.

- Thus we have drive $\underline{a}$ to zero!
- For a 3-D vector we would need 3 $\hat{e}_i$'s.
- When we consider a function, an infinite number of test functions will be needed to drive the error to zero. However we also need to increase the number of linearly independent functions in the trial functions such that we have a sufficient number of degrees of freedom.